

ON THE COFINALITY OF THE SMALLEST COVERING OF THE REAL LINE BY MEAGER SETS II

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ABSTRACT. We study the ideal of meager sets and related ideals.

1. INTRODUCTION

This paper continues the line of investigation started in [4] and strengthens the main result from [9].

Definition 1.1. Suppose that \mathcal{J} is a σ -ideal of subsets of 2^ω . Let

$$\text{add}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ \& } \bigcup \mathcal{A} \notin \mathcal{J}\}$$

and

$$\text{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ \& } \bigcup \mathcal{A} = 2^\omega\}.$$

Let \mathcal{M} and \mathcal{N} denote the ideals of meager and measure zero sets respectively. The goal of this paper is to study the cofinality of the cardinal $\text{cov}(\mathcal{M})$.

Recall that for a set $H \subseteq 2^\omega \times 2^\omega$ and $x, y \in 2^\omega$, $(H)_x = \{y \in 2^\omega : \langle x, y \rangle \in H\}$ and $(H)^y = \{x \in 2^\omega : \langle x, y \rangle \in H\}$.

The following definition is due to Reclaw ([9]).

Definition 1.2. A set $X \subseteq 2^\omega$ is an R set if for every Borel set $H \subseteq 2^\omega \times 2^\omega$, such that $(H)_x \in \mathcal{M}$ for all $x \in 2^\omega$,

$$\bigcup_{x \in X} (H)_x \neq 2^\omega.$$

We will use the following representation theorem for Borel sets with meager sections:

Lemma 1.3 (Fremlin [5]). *Suppose that $H \subseteq 2^\omega \times 2^\omega$ is a Borel set such that $(H)_x$ is meager for all x . Then there exists a sequence of Borel sets $\{G_n : n \in \omega\} \subseteq 2^\omega \times 2^\omega$ such that*

1. $(G_n)_x$ is a closed nowhere dense set for all $x \in 2^\omega$,
2. $H \subseteq \bigcup_{n \in \omega} G_n$.

PROOF For completeness we present a sketch of the proof here.

Let \mathcal{G} be the family of Borel subsets G of $2^\omega \times 2^\omega$ such that $(G)_x$ is open for every $x \in 2^\omega$.

1991 *Mathematics Subject Classification.* 04A20.

Key words and phrases. meager sets, cardinal invariants.

Research partially supported by the Israel Academy of Science, Basic Research Foundation.

Let \mathcal{J} be the σ -ideal of subsets of the plane generated by Borel sets F such that $(F)_x$ is closed nowhere dense for all x .

Consider the family Σ of subsets E of the plane such that there are two Borel sets G, H of which $G \in \mathcal{G}$, $H \in \mathcal{J}$ and $E \triangle G \subseteq H$. Clearly Σ contains all open sets and is closed under countable unions. We want to show that Σ is also closed under complements.

For a set $G \in \mathcal{G}$ let

$$G' = \{\langle x, y \rangle : y \text{ is an interior point of } 2^\omega \setminus (G)_x\}.$$

Note that $(2^\omega \times 2^\omega) \setminus (G \cup G')$ is a set whose vertical sections are closed and nowhere dense. It follows that in order to show that Σ is closed under complements it is enough to check that G' is a Borel set.

Let $\{U_n : n \in \omega\}$ be a recursive enumeration of a countable base for the family of open subsets of 2^ω .

Note that the following are equivalent:

1. $\langle x, y \rangle \in G'$,
2. $\exists n \left(y \in U_n \ \& \ \forall z \left(z \notin U_n \ \vee \ \langle x, z \rangle \notin G \right) \right) \ (\Pi_1^1)$,
3. $\exists n \left(y \in U_n \ \& \ \forall m \left(U_n \cap U_m = \emptyset \ \vee \ \exists z \left(z \in U_m \ \langle x, z \rangle \notin G \right) \right) \right) \ (\Sigma_1^1)$.

That shows that G' has a Δ_1^1 definition which means that it is a Borel set.

So Σ is a σ -algebra which contains every Borel set. Now if E is a Borel set with meager vertical sections then associated G must be empty. In particular, $E \in \mathcal{J}$. \square

2. R SETS AND THE COFINALITY OF $\text{cov}(\mathcal{M})$

In this section we will study R sets and show that R sets form an ideal closely related to the ideal of meager sets.

Definition 2.1. A set $X \subseteq 2^\omega$ has Rothberger's property (is a C'' set) if for every sequence of open covers of X , $\{\mathcal{G}_n : n \in \omega\}$ there exists a sequence $\{U_n : n \in \omega\}$ with $U_n \in \mathcal{G}_n$ such that $X \subseteq \bigcup_{n \in \omega} U_n$.

Rothberger's property is the topological version of strong measure zero. We have the following:

Theorem 2.2 (Fremlin, Miller [7]). *The following are equivalent:*

1. $X \subseteq 2^\omega$ has Rothberger's property,
2. X has strong measure zero with respect to every metric which gives X the same topology. \square

Let C'' be the collection of subsets of 2^ω which have Rothberger's property. It is easy to see that C'' is a σ -ideal.

Theorem 2.3. *The following conditions are equivalent:*

1. X is an R set,
2. for every Borel function $x \rightsquigarrow f^x \in \omega^\omega$ there exists a function $g \in \omega^\omega$ such that

$$\forall x \in X \ \exists^\infty n \ f^x(n) = g(n).$$

3. Every Borel image of X has Rothberger's property.

PROOF (1) \rightarrow (2). Suppose that $x \rightsquigarrow f^x \in \omega^\omega$ is a Borel mapping. Let $H = \{\langle x, h \rangle \in X \times \omega^\omega : \forall^\infty n \ h(n) \neq f^x(n)\}$. Clearly H is a Borel set with all $(H)_x$ meager and if $g \notin \bigcup_{x \in X} (H)_x$ then g has required properties.

(2) \rightarrow (3) Suppose that Y is a Borel image of X . Let $\{\mathcal{G}_n : n \in \omega\}$ be a sequence of open covers of Y . Suppose that $\mathcal{G}_n = \{G_n^m : m \in \omega\}$ for $n \in \omega$. For $y \in Y$ let $f^y \in \omega^\omega$ be the function defined as

$$f^y(n) = \min\{m : y \in G_n^m\} \text{ for } n \in \omega.$$

Mapping $y \rightsquigarrow f^y$ is Borel. Thus there exists a function $f \in \omega^\omega$ such that

$$\forall y \in Y \ \exists^\infty n \ f^y(n) = f(n).$$

Clearly the sequence $\{G_n^{f(n)} : n \in \omega\}$ has required properties.

(3) \rightarrow (2) Suppose that $x \rightsquigarrow f^x \in \omega^\omega$ is a Borel mapping. Clearly $Y = \{f^x : x \in X\}$ has Rothberger's property in ω^ω . Consider the families

$$\mathcal{G}_n = \{G_n^m : m \in \omega\} \text{ where } G_n^m = \{h \in \omega^\omega : h(n) = m\}.$$

The selector chosen for this sequence of coverings immediately gives us the function which meets every f^x in at least one point.

To get the function we are looking for we have to split ω into infinitely many pieces and apply the above construction to each one of them.

(2) \rightarrow (1) We will need several lemmas. To avoid repetitions let us define:

Definition 2.4. Suppose that $X \subseteq 2^\omega$. X is nice if for every Borel function $x \rightsquigarrow f^x \in \omega^\omega$ there exists a function $g \in \omega^\omega$ such that

$$\forall x \in X \ \exists^\infty n \ f^x(n) = g(n).$$

We will show first that:

Lemma 2.5. Suppose that X is nice. Then for every Borel function $x \rightsquigarrow \langle Y^x, f^x \rangle \in [\omega]^\omega \times \omega^\omega$ there exists $g \in \omega^\omega$ such that

$$\forall x \in X \ \exists^\infty n \in Y^x \ f^x(n) = g(n).$$

PROOF Suppose that a Borel mapping $x \rightsquigarrow \langle Y^x, f^x \rangle$ is given. Let y_n^x denote the n -th element of Y^x for $x \in X$. For every $x \in X$ define a function h^x as follows:

$$h^x(n) = f^x \upharpoonright \{y_0^x, y_1^x, \dots, y_n^x\} \text{ for } n \in \omega.$$

Since the mapping $x \rightsquigarrow h^x$ is Borel and functions h^x can be coded as elements of ω^ω there is a function h such that

$$\forall x \in X \ \exists^\infty n \ h^x(n) = h(n).$$

Without loss of generality we can assume that $h(n)$ is a function from an $n+1$ -element subset of ω into ω .

Define $g \in \omega^\omega$ in the following way. Choose inductively

$$z_n \in \text{dom}(h(n)) \setminus \{z_0, z_1, \dots, z_{n-1}\} \text{ for } n \in \omega.$$

Let g be any function such that $g(z_n) = h(n)(z_n)$ for $n \in \omega$.

We show that the function g has the required properties. Suppose that $x \in X$. Notice that the equality $h^x(n) = h(n)$ implies that

$$f^x(z_n) = g(z_n) \text{ and } z_n \in Y^x.$$

That finishes the proof since $h^x(n) = h(n)$ for infinitely many $n \in \omega$. \square

Lemma 2.6. *Suppose that X is nice. Then for every Borel mapping $x \rightsquigarrow f^x \in \omega^\omega$ there exists an increasing sequence $\{n_k : k \in \omega\}$ such that*

$$\forall x \in X \exists^\infty k \ f^x(n_k) < n_{k+1}.$$

PROOF Suppose that the lemma is not true and let $x \rightsquigarrow f^x$ be the witness. Without loss of generality we can assume that f^x is increasing for all $x \in X$. To get a contradiction we will define a Borel mapping $x \rightsquigarrow g^x \in \omega^\omega$ such that $\{g^x : x \in X\}$ is a dominating family. That will contradict the assumption that X is nice.

Define

$$g^x = \max\{\underbrace{f^x \circ f^x \circ \dots \circ f^x}_j(i) : i, j \leq n\} \text{ for } n \in \omega.$$

Suppose that $g \in \omega^\omega$ is an increasing function. By the assumption there exist $x \in X$ and k_0 such that

$$\forall k \geq k_0 \ f^x(g(k)) \geq g(k+1).$$

In particular,

$$\forall k \geq g(k_0) \ g(k) \leq g^x(k)$$

which finishes the proof. \square

Lemma 2.7. *Suppose that X is nice. Then for every Borel mapping $x \rightsquigarrow Y^x \in [\omega]^\omega$ there exists a set $Y = \{u_n : n \in \omega\}$ such that*

1. $u_{n+1} \geq u_n + 2$ for all n ,
2. $\forall x \in X \ |Y \cap Y^x| = \aleph_0$.

PROOF By applying 2.7, we can find an increasing function $f \in \omega^\omega$ such that

$$\forall x \in X \ |Y^x \setminus \{f(n) : n \in \omega\}| = \aleph_0.$$

Let $A_0 = \{2k : k \in \omega\}$ and $A_1 = \{2k+1 : k \in \omega\}$. Define Borel mapping $x \rightsquigarrow \langle Z^x, g^x \rangle \in [\omega]^\omega \times 2^\omega$ as follows: $Z^x = \text{dom}(g^x)$ and for $n \in \omega$,

$$g^x(n) = \begin{cases} 0 & \text{if } Y^x \cap (f(n), f(n+1)) \cap A_0 \neq \emptyset, \\ 1 & \text{if } Y^x \cap (f(n), f(n+1)) \cap A_1 \neq \emptyset \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that the first two conditions of this definition are not exclusive. We use either value when that happens.

By 2.7, there exists a function $h \in 2^\omega$ such that

$$\forall x \in X \exists^\infty n \in Z^x \ h(n) = g^x(n).$$

Define

$$Y = \bigcup_{n \in \omega} ((f(n), f(n+1)) \cap A_{h(n)}).$$

It is clear that Y has required properties. \square

We now return to the proof that (2) implies (1). Let $H \subseteq 2^\omega \times 2^\omega$ be a Borel set with all fibers $(H)_x$ meager. Using 1.3, find Borel sets $\{F^n : n \in \omega\}$ such that $H \subseteq \bigcup_{n \in \omega} F^n$ and $(F^n)_x$ is closed nowhere dense for $x \in 2^\omega$, $n \in \omega$.

For $x \in X$, define

$$s_n^x = \min \{s \in 2^{<\omega} : \forall t \in 2^{<n} \forall j \leq n [t \frown s] \cap (F^j)_x = \emptyset\} \text{ for } n \in \omega.$$

(where the minimum is taken with respect to some enumeration of $2^{<\omega}$).

By 2.6, there exists a sequence $\{n_k : k \in \omega\}$ such that

1. $n_{k+1} > \sum_{i=0}^k n_i$, for all k ,
2. $\forall x \in X \exists^\infty n |s_{n_k}^x| < n_{k+1}$.

For $x \in X$ let $Z^x = \{k : |s_{n_k}^x| < n_{k+1}\}$. By 2.5, there exists a sequence $\langle s_k : k \in \omega \rangle$ such that

$$\forall x \in X \exists^\infty k \in Z^x s_{n_k}^x = s_k.$$

Without loss of generality we can assume that $|s_k| < n_{k+1}$ for all k . Define mapping $x \rightsquigarrow Y^x$ by

$$Y^x = \{k \in Z^x : s_k = s_{n_k}^x\}.$$

Let Y be a set obtained by applying 2.7 to this family. Define

$$z = s_{l_0} \frown s_{l_1} \frown s_{l_2} \frown \dots, \text{ where } l_0 < l_1 < l_2 \dots \text{ is the increasing enumeration of } Y.$$

Note that if $l_{k+1} \in Y \cap Y^x$ then

$$|s_{l_0} \frown s_{l_1} \frown \dots \frown s_{l_k}| < \sum_{j \leq l_k} n_{l_j+1} < n_{l_k+2} \leq n_{l_{k+1}}$$

and

$$s_{l_{k+1}} = s_{n_{l_{k+1}}}^x.$$

In other words, $[s_{l_0} \frown s_{l_1} \frown \dots \frown s_{l_{k+1}}] \cap (F^i)_x = \emptyset$ for $i \leq l_{k+1}$. But that clearly implies that $z \notin \bigcup_{n \in \omega} (F^n)_x$, which finishes the proof. \square

As a corollary we get the following:

Theorem 2.8 (Reclaw [9]). *Every Luzin set is an R set.*

PROOF It is well known that every Borel image of a Luzin set has Rothberger's property. \square

Theorem 2.9. $\text{cf}(\text{cov}(\mathcal{M})) \geq \text{add}(R) \geq \text{add}(C'') \geq \text{add}(\mathcal{N})$.

PROOF We will start with the following lemma:

Lemma 2.10. *There exists a Borel set $H \subseteq 2^\omega \times 2^\omega$ with such that all $(H)_x$ are meager and for every set $A \in \mathcal{M}$ there exists $x \in 2^\omega$ such that $A \subseteq (H)_x$.*

PROOF For $z \in 2^\omega$ and $f \in \omega^\omega$ define a set

$$B(z, f) = \{t \in 2^\omega : \forall^\infty n \exists j \in [\tilde{f}(n), \tilde{f}(n+1)) t(j) \neq z(j)\}, \text{ where } \tilde{f}(n) = \sum_{k=0}^n f(k).$$

It is well known (see [2]) that the family $\{B(z, f) : z \in 2^\omega, f \in \omega^\omega\}$ is a basis of \mathcal{M} .

Fix a Borel surjection $t \rightsquigarrow \langle z_t, f_t \rangle \in 2^\omega \times \omega^\omega$ and let H be a set such that $(H)_t = B(z_t, f_t)$ for all $t \in 2^\omega$. \square

Suppose that $\mathcal{A} \subseteq \mathcal{M}$ is a family of meager sets of size $\text{cov}(\mathcal{M})$ which covers 2^ω . For $F \in \mathcal{A}$ let $x_F \in 2^\omega$ be such that $F \subseteq (H)_{x_F}$. Suppose that $\text{cf}(\text{cov}(\mathcal{M})) = \kappa$. It follows that $X = \{x_F : F \in \mathcal{A}\}$ is the union of κ many sets X_α of size smaller than $\text{cov}(\mathcal{M})$. Clearly each set $X_\alpha \in R$ and $X \notin R$. Thus $\kappa \geq \text{add}(R)$ which proves the first inequality.

The second inequality follows immediately from 2.3.

The third inequality is due to Carlson. We will prove it here for completeness. We will use the following fact (see [2] or [1]):

Theorem 2.11. *$\text{add}(\mathcal{N})$ is the smallest size of the family $F \subseteq \omega^\omega$ such that there is no function $S : \omega \longrightarrow [\omega]^{<\omega}$ with $|S(n)| \leq n$ for all n , such that*

$$\forall f \in F \forall^\infty n \ f(n) \in S(n). \quad \square$$

Suppose that $\{X_\alpha : \alpha < \kappa < \text{add}(\mathcal{N})\}$ is a family of C'' sets. We will show that $X = \bigcup_{\alpha < \kappa} X_\alpha$ is a C'' set.

Let $\{\mathcal{G}_n : n \in \omega\}$ be a sequence of open coverings of X . Assume that $\mathcal{G}_n = \{U_m^n : m \in \omega\}$ for $n \in \omega$. Let $r(n) = 1 + 2 + \dots + (n - 1)$. Define for $n \in \omega$ the family

$$\tilde{\mathcal{G}}_n = \left\{ \tilde{U}_s^n : s \in \omega^{[r(n), r(n+1))} \right\}, \text{ where } \tilde{U}_s^n = \bigcap_{j=r(n)}^{r(n+1)} U_{s(j)}^j.$$

Note that $\tilde{\mathcal{G}}_n$ is also a cover of X . By the assumption, for every $\alpha < \kappa$ there exists a function f_α such that $\{\tilde{U}_{f_\alpha(n)}^n : n \in \omega\}$ is a covering of X_α .

Since $\kappa < \text{add}(\mathcal{N})$, by 2.11, there exists a function $S : \omega \longrightarrow \omega^{<\omega}$ such that

$$\forall \alpha < \kappa \forall^\infty n \ f_\alpha(n) \in S(n).$$

Without loss of generality we can assume that $S(n)$ consists of at most n sequences of length n . Let f be a function which agrees at least once on the n -element interval $[r(n), r(n+1))$ with each of the n functions in $S(n)$. Clearly $\{\tilde{U}_{f(n)}^n : n \in \omega\}$ is the covering of X we were looking for. \square

Remarks and questions

- (1) Is $\text{cf}(\text{cov}(\mathcal{M})) \geq \text{add}(\mathcal{M})$? In other words, is $\text{cf}(\text{cov}(\mathcal{M})) \geq \mathfrak{b}$?
- (2) The first inequality in 2.9 also holds when we replace category by measure. Unfortunately we do not know if R sets defined for measure form an ideal. (For more see [8] and [3]).
- (3) It is consistent that $\text{add}(\mathcal{M}) > \text{add}(C'')$ ([6]).
- (4) Is it consistent that $\text{add}(C'') > \text{add}(\mathcal{N})$? $\text{add}(R) > \text{add}(\mathcal{N})$?

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